Quantum Analysis of Nested Search Problems

with Applications in Cryptanalysis

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Outline

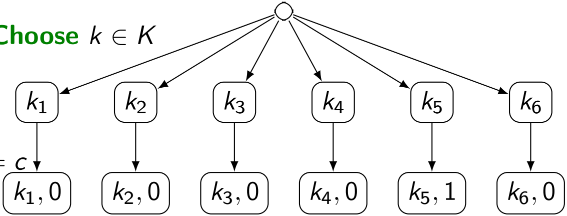
Search problems (and search trees)

- Each edge "$\to$" is a computing step
- Branching $=$ making a choice

Example: search k such that $E_k(p) = c$.

Choose $k \in K$

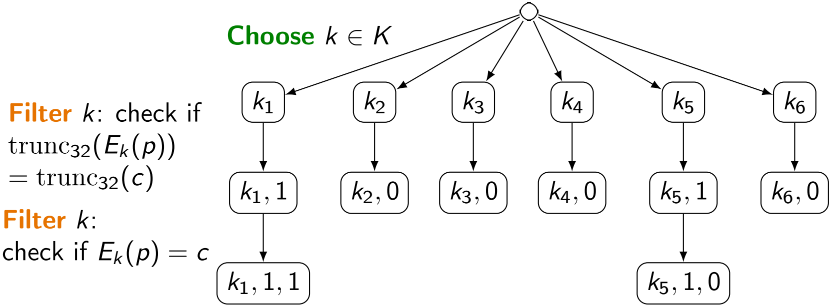
Filter k:
check if $E_k(p) = c$



$$\text{Time} = |K| \times \text{Evaluate } E_k$$

Complex search problems

In cryptanalysis, we consider complex search problems.
Example: search k such that $E_k(p) = c$, in two steps.



$$\text{Time} = |K| \times \text{Evaluate } (\mathrm{trunc}_{32} \circ E_k) + \frac{|K|}{2^{32}} \times \text{Evaluate } E_k$$

Search problems everywhere

In symmetric cryptanalysis:

- differential, linear attacks with dynamic key-guessing
- impossible differential, zero-correlation attacks
- boomerang attacks
- MITM and DS-MITM attacks . . .

are all (at least partially) exploring search trees with **choices**, **filtering** and **backtracking**.

$\Longrightarrow$ We want to turn them into **quantum attacks** using **quantum search**.

From search to quantum search

Amplitude amplification (QAA) starts from a quantum algorithm

$$\mathcal{A}\left|0\right\rangle = \alpha \underbrace{\left|\psi\right\rangle}_{\text{Good}} \left|1\right\rangle + \sqrt{1-\alpha^2} \underbrace{\left|\phi\right\rangle}_{\text{Bad}} \left|0\right\rangle$$

succeeding with probability α^2 and **amplifies** this to $\simeq 1$ with $\mathcal{O}\left(1/\alpha\right)$ iterates of $\mathcal{A}$. What it looks like:

$$\underbrace{(\mathcal{A}O_0\mathcal{A}^\dagger O)}_{\text{QAA iterate}} \cdots (\mathcal{A}O_0\mathcal{A}^\dagger O)\mathcal{A}\left|0\right\rangle$$

After t iterates:

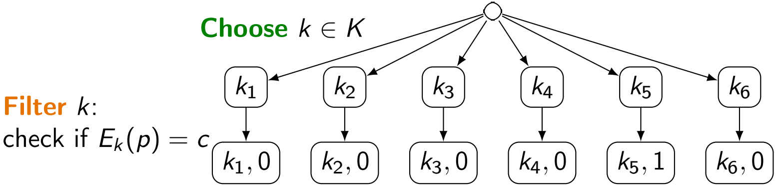
$$\nu\left|\psi\right\rangle\left|1\right\rangle + \sqrt{1-\nu^2}\left|\phi\right\rangle\left|0\right\rangle$$

where

$$\nu = \sin\left[(2t+1)\arcsin\left[\alpha\right]\right]$$

Brassard, Høyer, Mosca, Tapp, "Quantum amplitude amplification and estimation", Contemp. math. 2002

From search to quantum search (ctd.)



By iterating the classical operation "**choose** and **filter**" until it succeeds, classical search succeeds in time:

$$\text{Time} = |K| \times (\textbf{Choose} + \textbf{Filter})$$

By iterating a **quantum algorithm** for "**choose** and **filter**", QAA succeeds in time:

$$\text{Time} = \sqrt{|K|} \times (\textbf{Choose} + \textbf{Filter})$$

QAAs all the way down

QAA is a quantum algorithm, so we can run a QAA inside a QAA.

Folklore conversion lemma

- Any classical nested search algorithm can be converted into a quantum search algorithm.
- The quantum complexity is obtained by replacing "Iterations" with "$\sqrt{\text{Iterations}}$"

The problem(s)

1. Rewrite classical attacks in a way that facilitates the "conversion" (partially solved)

2. Compute **exact** complexities for the quantum algorithms.
   - So far handled on a case-by-case basis with lots of technical analysis.
   - Our goal is to externalize this work using a generic framework & analysis.

Results

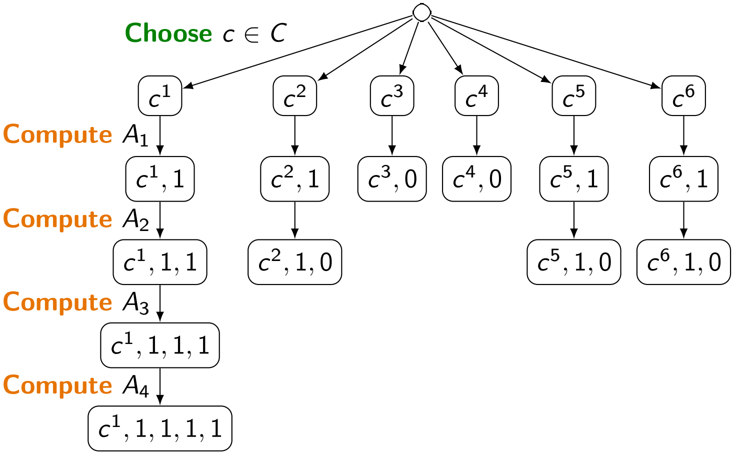
Two quantum algorithms for nested search:

- search with early aborts
- search with backtracking

with exact complexity analysis and optimizations of previous attacks.

Search with Early Aborts

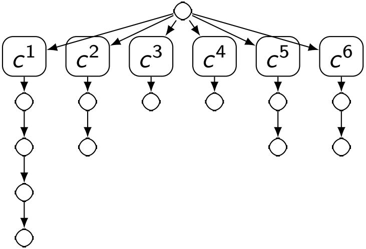
Restricted setting: search with early aborts



- Start from a single choice space C
- Search $c \in C$ that passes all filters: $A_4 \circ A_3 \circ A_2 \circ A_1(c) = 1$
- Many problems where we can "try a few bits first"

Generic idea

- Assume that A_i are implemented by quantum algorithms $\mathcal{A}_i$
- Each A_i creates a new "flag"
- Let β_i^2 = prob. of passing step i if we passed step $i-1$



$\beta_1^2 = 4/6$
$\beta_2^2 = 1/4$
$\beta_3^2 = 1/1$
$\beta_4^2 = 1/1$

There are ℓ levels. We create ℓ quantum algorithms $\mathcal{B}_i$ such that:

$$\mathcal{B}_i |0\rangle = \nu_i^2 |\psi_i\rangle |1\rangle + \sqrt{1-\nu_i^2} |\phi_i\rangle |0\rangle$$

where ν_i^2 is the **probability of success** and $|\psi_i\rangle$ is the unif. superposition of choices passing step i.

Generic idea (ctd.)

$\mathcal{B}_1$ is a QAA with k_1 iterates. The amplified algorithm $\mathcal{A}_1'$ does:

1. choose $c \in C$ unif. at random
2. apply $\mathcal{A}_1$

$$\beta_1^2 \left|\psi_1\right\rangle \left|1\right\rangle + \sqrt{1 - \beta_1^2} \left|\phi_1\right\rangle \left|0\right\rangle$$

So it has a success probability β_1^2 which we amplify with k_1 iterates to ν_1^2:

$$\nu_1 = \sin\left[(2k_1 + 1) \arcsin \beta_1\right]$$

Generic idea (ctd.)

We define $\mathcal{B}_i$ by a QAA with k_i iterates, which amplifies the algorithm:

1. Apply $\mathcal{B}_{i-1}$
2. Apply $\mathcal{A}_i$

By assumption:

$$\mathcal{A}_i \mathcal{B}_{i-1} |0\rangle = \nu_{i-1} \beta_{\mathrm{i}} |\psi_i\rangle |1\rangle + (\ldots) |0\rangle$$

so after amplifying with k_i iterates we get:

$$\nu_i = \sin\left[(2k_i + 1) \arcsin\left[\beta_{\mathrm{i}} \nu_{i-1}\right]\right]$$

Bounding the probability

Example: for 3 levels with k_1, k_2, k_3 iterates:

$$\sin\left[(2k_3+1)\arcsin\left[\beta_3 \sin\left[(2k_2+1)\arcsin\left[\beta_2 \sin\left[(2k_1+1)\arcsin\beta_1\right]\right]\right]\right]\right]$$

$$\mathrm{Time} = (2k_3+1)\Big((2k_2+1)\Big((2k_1+1)\mathrm{Time}(\mathcal{A}_1)+\mathrm{Time}(\mathcal{A}_2)\Big)+\mathrm{Time}(\mathcal{A}_3)\Big)$$

- For small x, $\sin x \simeq x \simeq \arcsin x$
- So if all probabilities remain "small":
  $$\nu_3 \simeq (2k_3+1)\beta_3(2k_2+1)\beta_2(2k_1+1)\beta_1$$
- Of course this shouldn't be "too small": the balance lies at $\mathcal{O}(1/\ell)$.

Bounding the probability (ctd.)

If $\forall i, \prod_{j=1}^{i} \left((2k_j+1)^2 \beta_j^2\right) \leq \frac{4}{\pi^2 \ell}$, then: $\nu_\ell^2 \geq \frac{1}{5} \prod_{j=1}^{\ell} \left((2k_j+1)^2 \beta_j^2\right)$.

To be read as:

If intermediate probabilities of success ($\simeq \prod_{j=1}^{i} \left((2k_j+1)^2 \beta_j^2\right)$) are of order $\frac{1}{\ell}$ then the final probability of success is of order $\frac{1}{\ell}$.

- We can amplify on top of this using $\mathcal{O}\left(\sqrt{\ell}\right)$ iterates, to a constant prob. of success.
- ℓ levels of QAA multiply the complexity by $\sqrt{\ell}$ (w.r.t. the exact square root)

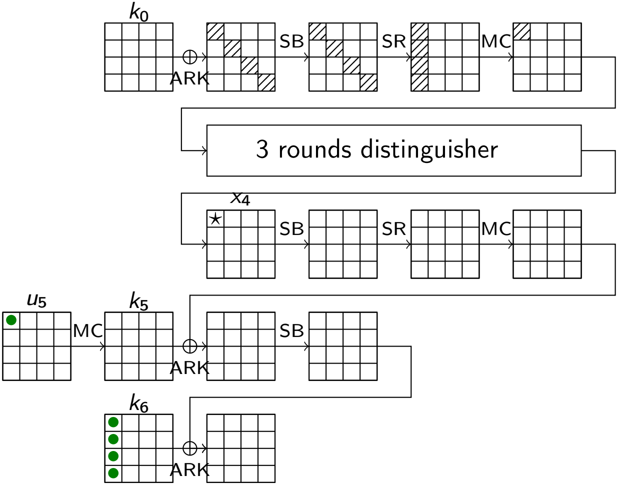
Applications

1. we know (good bounds on) the β_i
$\Longrightarrow$ minimize numerically Complexity$(\mathcal{B}_\ell)/\nu_\ell^2$ in function of k_i.

2. we don't know anything on the β_i
$\Longrightarrow$ variable-time QAA (in the paper)

Search with Backtracking

Example: 6-round AES Square attack



- Encrypt multiple structures of 2^{32} plaintexts (main diagonal varies)
- **Find** $u_5[0], k_6[0, 1, 2, 3]$ s.t. x_4 is balanced for all structures

Example (ctd.)

For each key guess we evaluate a sum over all ciphertexts:

$$\bigoplus_i S^{-1}\left(u_5[0] \oplus a_0 S^{-1}(c_i[0] \oplus k_6[0]) \oplus a_1 S^{-1}(c_i[1] \oplus k_6[1])\right.$$
$$\left. \oplus\, a_2 S^{-1}(t_2 \oplus k_6[2]) \oplus a_3 S^{-1}(s_2 \oplus k_6[3])\right)$$

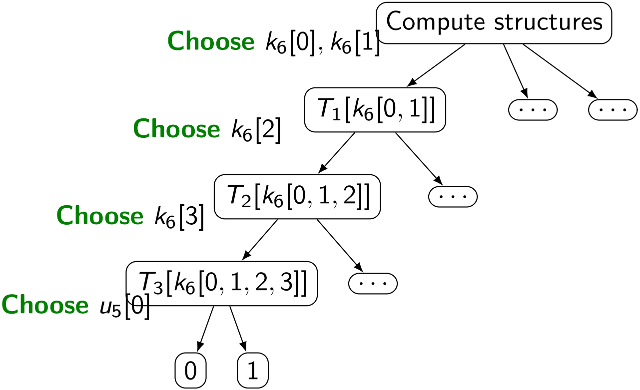
Partial sums method: store intermediate values in tables counting their number of occurrences.

- **Choose** $k_6[0], k_6[1]$. For each ciphertext c_i, compute: $(t_1, t_2, t_3) = a_0 S^{-1}(c_i[0] \oplus k_6[0]) \oplus a_1 S^{-1}(c_i[1] \oplus k_6[1]), c_i[2], c_i[3]$ and store occurrences in $T_1[k_6[0, 1]]$.
- **Choose** $k_6[2]$. For each 3-byte value (t_1, t_2, t_3), compute: $t_1 \oplus a_2 S^{-1}(t_2 \oplus k_6[2]), t_3$ and store occurrences in $T_2[k_6[0, 1, 2]]$.
- **Choose** $k_6[3]$. For each 2-byte value (s_1, s_2), compute $s_1 \oplus a_3 S^{-1}(s_2 \oplus k_6[3])$ and store occurrences in $T_3[k_6[0, 1, 2, 3]]$
- **Choose** $u_5[0]$. Compute final sum.

Ferguson, Kelsey, Lucks, Schneier, Stay, Wagner, Whiting, "Improved cryptanalysis of Rijndael", FSE 2000

Partial sums as a tree search

Example: Square attack on 6-round AES = search with backtracking.



$$\text{Time} = 2^{16} \times \left(\underbrace{\text{Compute } T_1}_{2^{32}} + 2^8 \times \left(\underbrace{T_2}_{2^{16}} + 2^8 \times \left(\underbrace{T_3}_{2^8} + 2^8 \times \underbrace{\text{sum}}_{2^8} \right) \right) \right)$$

General case

- **Tree search with backtracking** = sequence of **choices**, **filtering** and **post-processing** steps
  [Disclaimer: omitting the **filtering** in next slides]
- We must **return to previous states** to amortize the cost of each step
- Contains the attacks on AES of **[BNS19]** (Square and DS-MITM)

Bonnetain, Naya-Plasencia, S., "Quantum security analysis of AES", ToSC 2019

Sketch of the generic algorithm

- Consider a sequence of **choices** and algorithms $\mathcal{A}_i$: $\mathcal{A}_i$ depends on choice c_i and updates workspace
- A **single solution path** $c_1, \ldots, c_\ell$ s.t.: $\mathcal{A}_\ell \circ \cdots \circ \mathcal{A}_1(c_1, \ldots, c_\ell) = 1$
- New parameters: α_i^2 = probability, if $c_1, \ldots, c_{i-1}$ is the good path, that c_i extends it.

Before:

- $\mathcal{B}_i$ produces choices that pass test i
- $\mathcal{B}_i$ calls $\mathcal{B}_{i-1}$ and $\mathcal{A}_i$
- $\mathcal{B}_\ell$ solves the problem

Now:

- $\mathcal{B}_i$, **starting from the right subpath**, produces the entire solution
- $\mathcal{B}_i$ calls $\mathcal{B}_{i+1}$ and $\mathcal{A}_i$
- $\mathcal{B}_1$ solves the problem

Good news

The QAAs are nested in a reverse order, but the recursion formula is the same:

$$\nu_i = \sin\left[(2k_i + 1)\arcsin\left[\alpha_i \nu_{i+1}\right]\right]$$

It suffices to know estimates on α_i to obtain:

- an analytic formula for the success prob. and complexity (not always the best)
- a numerical optimization (always at least improving previous computations)

Results

Example: AES 6-round Square attack

- **[BNS19]**: $2^{44.73}$ AES S-Boxes, success prob. 1
- Analytic: $2^{48.70}$, success prob. 0.5
- Optimized: $2^{44.70}$, success prob. 0.94
- Square root of iterations: $2^{44.05}$

Example: AES-256 8-round DS-MITM

- **[BNS19]**: $2^{136.17}$ AES S-Boxes, success prob. 0.73
- Analytic: $2^{134.23}$, success prob. 0.5×0.73
- Optimized: $2^{132.07}$, success prob. 0.95×0.73
- Square root: $2^{131.07}$

Conclusion

The setting of our algorithms (balanced search trees, known parameters) captures most applications of QAA in cryptanalysis.

- Our formulas may be used to bound the complexities
- Numerical optimisation performs even better
- The final complexity will be **very** close to the “basic square root”

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